Linear density perturbation in relativistic and Brans-Dicke cosmologies

1977 J. Phys. A: Math. Gen. 10 189

(http://iopscience.iop.org/0305-4470/10/2/008)

J. Phys. A: Math. Gen., Vol. 10, No. 2, 1977. Printed in Great Britain. © 1977

Linear density perturbation in relativistic and Brans–Dicke cosmologies

Nikhilendu Bandyopadhyay
Satyendranath Bose Institute of Physical Sciences, Calcutta University, Calcutta 9, India

Received 7 June 1976, in final form 11 October 1976

Abstract. Linear density perturbation equation for general relativistic cosmologies using the homogeneous isotropic Robertson–Walker metric and the anisotropic Bianchi-I metric for the unperturbed model has been derived in a simple and rather general manner. The method has then been applied to consideration of the fate of density perturbation in the Brans–Dicke model of the universe. Two coupled differential equations connecting variations in the density ρ and the scalar field ϕ of the model have been obtained. An approximate solution has been sought and a power law growth of the fluctuation in density, $\delta\rho$, with time has been obtained.

1. Introduction

In this paper a simplified, general way of deducing the linear density perturbation equations in general relativisitic cosmologies is proposed, using both the isotropic, homogeneous Robertson–Walker metric and the anisotropic, homogeneous Bianchi-I metric for the unperturbed universe. A discussion of the fate of density perturbation in Brans–Dicke cosmology follows.

2. Linear density perturbation equation in general relativistic universes

We first consider the case where the unperturbed universe is homogeneous and isotropic with the line element

$$ds^2 = dt^2 - \frac{R_0^2(t)}{(1+\frac{1}{4}kr^2)^2}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{2.1}$$

where units have been chosen such that the velocity of light $c = 1$, $k = \pm 1, 0$, and $R_0(t)$ is the cosmic scale factor, the subscript 0 on R referring to the unperturbed value.

In the perturbed universe we assume that there is still streamline flow so that we can introduce a co-moving system of reference in which the line element can be written as

$$ds^2 = dt^2 + 2g_{i0}\, dt\, dx^i + g_{ik}\, dx^i\, dx^k, \tag{2.2}$$

where the g_{0i} terms make allowance for the presence of vorticity and/or a translational velocity relative to the frame of isotropy of the unperturbed universe as well as for the non-geodesic nature of the world lines. (In our discussion Latin indices will run from 1 to 3, while Greek indices will run from 0 to 3, the index zero corresponding to the time coordinate.)

The smallness of the perturbation may be taken to mean that each of the quantities

$$v_i = g_{0i}; \qquad p_{,i}; \qquad \rho_{,i}; \qquad g_{ik} + \frac{R_0^2}{(1+\frac{1}{4}kr^2)^2}\delta_{ik}$$

(the last form presupposes that the space coordinates x^i are orthogonal Cartesian) as well as their derivatives are all small so that only terms linear in them will be considered in our calculations. These quantities vanish in the unperturbed universe.

With the perfect fluid energy–momentum tensor

$$T^{\mu\nu} = (p+\rho)v^\mu v^\nu - pg^{\mu\nu}$$

the divergence relation $T^{\mu\nu}{}_{;\mu} = 0$ gives the following two equations:

$$\dot\rho = -(p+\rho)\Theta \tag{2.3}$$

$$\dot v^\nu = p_{,\mu}(g^{\mu\nu} - v^\mu v^\nu)/(p+\rho) \tag{2.4}$$

where $\Theta = v^\mu{}_{;\mu}$ is the expansion which has the value $3\dot R_0 R_0^{-1}$ for the unperturbed universe. In the perturbed universe we set $\Theta = 3\dot R R^{-1}$. R, so defined, will differ slightly from R_0.

With the line element (2.2) the above two equations give

$$\dot\rho = -3(p+\rho)\dot R R^{-1} \tag{2.5}$$

and

$$\dot g_{0i} = (p_{,i} - \dot p g_{0i})/(p+\rho). \tag{2.6}$$

Now to study the fate of density perturbation we consider variation of the Raychaudhuri equation

$$\dot\Theta + \tfrac{1}{3}\Theta^2 - \dot v^\alpha{}_{;\alpha} + 2(\sigma^2 - \omega^2) + 4\pi G(\rho + 3p) = 0 \tag{2.7}$$

where σ and ω are the shear and vorticity scalars, respectively. In the unperturbed universe we have $\sigma = \omega = 0$, and so ω^2 and σ^2 can be neglected as terms of second order.

Also we have, using (2.4), (2.6) and (2.2):

$$\dot v^\alpha{}_{;\alpha} = [\nabla^2 p - (\dot p/\sqrt{-g})(g_{i0}g^{ik}\sqrt{-g})_{,k}]/(p+\rho) \tag{2.8}$$

where ∇^2 is the Laplacian for the three-space metric

$$\mathrm{d}l^2 = g_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k.$$

Now, if we introduce the condensation parameter s by

$$\rho = \rho_0(1+s) \tag{2.9}$$

and assume a linear equation of state

$$p = \alpha\rho \qquad (\alpha = \text{constant}) \tag{2.10}$$

we have using (2.5)

$$\delta\Theta = -\dot s/(1+\alpha) \qquad \delta\dot\Theta = -\ddot s/(1+\alpha) \tag{2.11}$$

and also from (2.8)

$$\delta(\dot v^\alpha{}_{;\alpha}) = \frac{\alpha}{1+\alpha}\nabla^2 s + \frac{3\alpha}{\sqrt{-g}}\frac{\dot R_0}{R_0}(g_{i0}g^{ik}\sqrt{-g})_{,k}. \tag{2.12}$$

Thus variation of the Raychaudhuri equation gives, on using (2.9), (2.10), (2.11) and (2.12),

$$\ddot{s}+2\frac{\dot{R}_0}{R_0}\dot{s}+\alpha\nabla^2 s+\frac{3\alpha(1+\alpha)}{\sqrt{-g}}\frac{\dot{R}_0}{R_0}(g_{i0}g^{ik}\sqrt{-g})_{,k}-4\pi\rho_0 G(1+3\alpha)(1+\alpha)s=0. \tag{2.13}$$

Again, (2.6) gives, on integration,

$$g_{0i}=\frac{\alpha}{1+\alpha}R_0^{-3\alpha}\int R_0^{3\alpha}s_{,i}\,\mathrm{d}t+f_i R_0^{-3\alpha} \tag{2.14}$$

where the f_i are functions of space coordinates alone. The first term in g_{0i} is a gradient $\phi_{,i}$ and can be removed by an infinitesimal transformation $t'=t+\phi(x^i, t)$ which would not change the form of the metric. The second term may give rise to vorticity and in (2.13) gives rise to a term:

$$\alpha\frac{\dot{R}_0}{R_0}R_0^{-3\alpha}g^{ik}f_{i,k}$$

which has a time dependence of approximately $t^{-3/2}$ for the radiation universe in which $R_0\sim t^{1/2}$, and a zero contribution for the dust universe in which $\alpha=0$. Locally $f_{i,k}+f_{k,i}$ may be made to vanish by a suitable transformation and in any case the term decreases rapidly with time. We shall therefore disregard this term in our subsequent discussion.

We are, therefore, led to the equation

$$\ddot{s}+2\frac{\dot{R}_0}{R_0}\dot{s}+\alpha\nabla^2 s-4\pi G\rho_0(1+3\alpha)(1+\alpha)s=0. \tag{2.15}$$

This is the well known linear density perturbation equation. It has been obtained previously by a number of authors—for example, Bonnor (1957), Silk and Brecher (1969) and Irvine (1965)—with somewhat less generality. The consequences of this equation have been studied extensively in the literature (Liang 1976) while non-linear perturbations have been studied by Field and Shepley (1968). We do not intend to recapitulate these results but make a few remarks for the case $\alpha=0$. In this case equation (2.15) becomes:

$$\ddot{s}+2\frac{\dot{R}_0}{R_0}\dot{s}-4\pi G\rho_0 s=0.$$

Disregarding for a moment the term in $\dot{s}$, the equation shows a monotonic increase of s but at a rate too slow to account for the formation of galaxies. The term in $\dot{s}$ is similar to a damping term and in an expanding universe (i.e., $\dot{R}_0/R_0$ is positive) would slow down still further this growth of s. These conclusions, to some extent, contradict the conjecture of Hawking (1966) that in a pressureless universe the fate of a condensation would depend critically on the curvature parameter.

The problem of density perturbation in the Bianchi-IX and Bianchi-I universes has been studied by Hu and Regge (1972) and by Perko *et al* (1972) respectively. We content ourselves with deriving the linear density perturbation equation for the Bianchi type-I universe by the above method. In this case the line element for the background homogeneous universe is

$$\mathrm{d}s^2=\mathrm{d}t^2-R_{10}^2(t)\,\mathrm{d}x^2-R_{20}^2(t)\,\mathrm{d}y^2-R_{30}^2(t)\,\mathrm{d}z^2 \tag{2.16}$$

(suffix 0 denoting unperturbed values) follows the same line of reasoning as in the previous case. Here Θ in (2.3) will be given, in the unperturbed universe, by

$$\Theta = \theta_{10} + \theta_{20} + \theta_{30} = \frac{\dot{R}_{10}}{R_{10}} + \frac{\dot{R}_{20}}{R_{20}} + \frac{\dot{R}_{30}}{R_{30}}.$$

In the perturbed universe, for which the line element is again assumed to be of the form (2.2), Θ is put equal to $\theta_1 + \theta_2 + \theta_3$ where $\theta_i = \dot{R}_i/R_i$. The R_i so defined will differ very little from the R_{i0}.

Here for the unperturbed model we have, to our order of approximation,

$$\omega = 0, \qquad 3\sigma^2 = \Theta^2 - 3(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1) \tag{2.17}$$

whereas the field equations

$$R_{\mu\nu} = -8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu})$$

give, specializing for the case of a dust universe,

$$\begin{aligned} &\dot{\Theta} + \theta_1^2 + \theta_2^2 + \theta_3^2 = -4\pi G\rho \qquad && \dot{\theta}_1 + \theta_1\Theta = -4\pi G\rho \\ &\dot{\theta}_2 + \theta_2\Theta = -4\pi G\rho && \dot{\theta}_3 + \theta_3\Theta = -4\pi G\rho. \end{aligned} \tag{2.18}$$

From (2.18) and (2.3) with $p = 0$, we have

$$-\rho\ddot{\rho} + 2\dot{\rho}^2 = \tfrac{3}{2}\rho^3$$

with the integral

$$\rho = 4/(3t^2 + 4At) \tag{2.19}$$

where A is a constant of integration, and where the condition $\rho \to \infty$ as $t \to 0$ has been used.

Also from (2.17) and (2.18) it follows that

$$3\sigma^2 = A^2\rho^2. \tag{2.20}$$

Using (2.3), (2.17), (2.20) in the variation of the Raychaudhuri equation for the dust universe gives

$$\ddot{s} - \frac{2}{3}\frac{\dot{\rho}}{\rho}\dot{s} - (\tfrac{4}{3}A^2\rho^2 - 4\pi G\rho)s = 0 \tag{2.21}$$

which gives, on using (2.19)

$$\ddot{s} + \frac{4}{3t}\left(\frac{3t + 2A}{3t + 4A}\right)\dot{s} - (\tfrac{4}{3}A^2\rho^2 + 4\pi G\rho)s = 0. \tag{2.22}$$

This equation has been deduced by Johri (1972) using the tetrad formalism.

3. Fate of density perturbation in Brans–Dicke cosmology

We start our discussion of the fate of density perturbation in Brans–Dicke cosmology using the extended Raychaudhuri equation as given by Banerji (1974):

$$3\frac{\ddot{R}}{R}-\dot{u}^{\mu}{}_{;\mu}+2(\sigma^2-\omega^2)=-\frac{8\pi}{\phi}\left(\frac{2+\Omega}{3+2\Omega}\rho+\frac{3(1+\Omega)}{3+2\Omega}p\right)+\frac{\Omega}{\phi^2}\dot{\phi}^2+\frac{\ddot{\phi}}{\phi}-\frac{\phi_\mu \dot{u}^\mu}{\phi}, \tag{3.1}$$

Ω being the coupling constant between the tensor and scalar fields.

The shear and vorticity terms are as before of the second order and left out of consideration in our order of approximation. In view of our discussions in § 2, we obtain, for small density perturbation:

$$\begin{aligned}\frac{\ddot{s}}{1+\alpha}&+\frac{2}{1+\alpha}\dot{s}\frac{\dot{R}_0}{R_0}+\frac{\alpha}{1+\alpha}\nabla^2 s\\ &=\frac{8\pi}{\phi}\left(\frac{2+\Omega}{3+2\Omega}+\frac{3\alpha(1+\Omega)}{3+2\Omega}\right)\rho_0 s-\frac{8\pi}{\phi^2}\left(\frac{2+\Omega}{3+2\Omega}+\frac{3\alpha(1+\Omega)}{3+2\Omega}\right)\rho_0\,\delta\phi\\ &\quad+\frac{2\Omega}{\phi^3}\dot{\phi}^2\,\delta\phi-\frac{2\Omega}{\phi^2}\dot{\phi}\,\delta\dot{\phi}-\frac{\delta\ddot{\phi}}{\phi}+\delta\left(\frac{\phi_\mu\dot{u}^\mu}{\phi}\right).\end{aligned} \tag{3.2}$$

We now specialize to the case of the dust universe with $p=0$. The above equation then becomes

$$\ddot{s}+2\frac{\dot{R}_0}{R_0}\dot{s}=\frac{8\pi}{\phi}\frac{2+\Omega}{3+2\Omega}\rho_0 s-\frac{8\pi}{\phi^2}\frac{2+\Omega}{3+2\Omega}\rho_0\,\delta\phi+\frac{2\Omega}{\phi^3}\dot{\phi}^2\,\delta\phi-\frac{2\Omega}{\phi^2}\dot{\phi}\,\delta\dot{\phi}+\frac{\ddot{\phi}}{\phi^2}\,\delta\phi-\frac{\delta\ddot{\phi}}{\phi}. \tag{3.3}$$

Again, using the following result of Brans–Dicke theory:

$$\phi^{\alpha}{}_{;\alpha}=8\pi T/(3+2\Omega),$$

and using the relation $\sqrt{-g}=R^3$, we have for the dust universe

$$\delta\ddot{\phi}+\frac{\dot{\phi}}{\sqrt{-g}}\left(g^{ik}g_{0k}\sqrt{-g}\right)_{,i}+3\frac{\dot{R}_0}{R_0}\,\delta\dot{\phi}-9\frac{\dot{R}_0}{R_0}-9\frac{\dot{R}_0}{R_0^2}\dot{\phi}\,\delta R+\frac{3}{R_0}\dot{\phi}\,\delta\dot{R}=\frac{8\pi\rho_0}{3+2\Omega}s. \tag{3.4}$$

We now use the following simple model of the Brans–Dicke dust universe (Dicke 1962)

$$R_0=at^{2(1+\Omega)/(3\Omega+4)}\qquad \rho_0=bt^{-6(1+\Omega)/(3\Omega+4)}\qquad \dot{\phi}=At^{-(3\Omega+2)/(3\Omega+4)} \tag{3.5}$$

where $a^3b=\rho_0R_0^3$ and $A=[8\pi\rho_{00}/(2\Omega+3)]t_0^{6(1+\Omega)/(3\Omega+4)}$; ρ_{00} being the average unperturbed density at the time t_0.

We then have from (3.3) and (3.5)

$$\begin{aligned}\ddot{s}&+\frac{4(1+\Omega)}{3\Omega+4}\frac{1}{t}\dot{s}-\frac{16\pi(2+\Omega)b}{(3\Omega+4)(3+2\Omega)A}\frac{1}{t^2}s\\ &=-\frac{2}{(3\Omega+4)A}t^{-2/(3\Omega+4)}\,\delta\ddot{\phi}-\frac{8\Omega}{(3\Omega+4)^2A}t^{-3(\Omega+2)/(3\Omega+4)}\,\delta\dot{\phi}\\ &\quad-\left(\frac{32\pi(2+\Omega)b}{(3\Omega+4)^2(3+2\Omega)A^2}+\frac{16\Omega}{(3\Omega+4)^3}\right)t^{-(6\Omega+10)/(3\Omega+4)}\,\delta\phi\end{aligned} \tag{3.6}$$

and from (3.4) and (3.5) and in view of our discussion in § 2 about g_{0i}

$$\delta\ddot{\phi}+\frac{4(1+\Omega)}{3\Omega+4}\frac{1}{t}\delta\dot{\phi}=At^{-(3\Omega+2)/(3\Omega+4)}\dot{s}-\left(\frac{4(1+\Omega)}{3\Omega+4}A-\frac{8\pi b}{3+2\Omega}\right)t^{-6(1+\Omega)/(3\Omega+4)}s. \tag{3.7}$$

To solve for s one must solve the two coupled equations (3.6) and (3.7), but an exact solution seems quite formidable. Nevertheless we can attempt to find an approximate solution in the following manner.

The last term on the right-hand side of equation (3.6) has the highest negative power of time and so in attempting an approximate solution we replace equation (3.6) by the following:

$$\ddot{s}+\frac{4(1+\Omega)}{3\Omega+4}\frac{1}{t}\dot{s}-\frac{16\pi(2+\Omega)b}{(3\Omega+4)(3+2\Omega)A}\frac{1}{t^2}s$$
$$=-\frac{2}{(3\Omega+4)A}t^{-2/(3\Omega+4)}\delta\ddot{\phi}-\frac{8\Omega}{(3\Omega+4)^2A}t^{-3(\Omega+2)/(3\Omega+4)}\delta\dot{\phi}. \tag{3.8}$$

Solving the coupled equations (3.7) and (3.8) (neglecting unity with respect to Ω in the coefficient of $\delta\dot{\phi}$ in equation (3.7)) we have

$$\ddot{s}+\frac{2(2\Omega+3)}{3\Omega+4}\frac{1}{t}\dot{s}-\left(\frac{8(1+\Omega)}{(3\Omega+4)^2}-\frac{16\pi(3+\Omega)b}{(3\Omega+4)(3+2\Omega)A}\right)\frac{1}{t^2}s=0 \tag{3.9}$$

which is of the form

$$\ddot{s}+\frac{P}{t}\dot{s}-\frac{Q}{t^2}s=0 \tag{3.10}$$

with obvious substitutions.

To solve equation (3.10) we try a solution $s\sim t^{\xi}$. Thus ξ satisfies

$$\xi^2+(P-1)\xi-Q=0 \tag{3.11}$$

The two solutions of the quadratic equation (3.11) are

$$\begin{aligned}\xi_1&=\tfrac{1}{2}\{-(P-1)+[(P-1)^2+4Q]^{1/2}\}\\ \xi_2&=\tfrac{1}{2}\{-(P-1)-[(P-1)^2+4Q]^{1/2}\}.\end{aligned} \tag{3.12}$$

We note that $\xi_1>0$ and $\xi_2<0$. Thus the condensation s has scope for growth in the dust-filled Brans–Dicke universe. But the power-law growth $s\sim t^{\xi_1}$ is definitely too low a rate to account for galaxy formation so long as the initial perturbations are assumed to result from statistical fluctuation of density.

Acknowledgment

The author is indebted to Professor A K Raychaudhuri, Professor of Physics, Presidency College, Calcutta, for suggesting the problem and extending helpful comments.

References

Banerji S 1974 *Phys. Rev.* D **9** 877

Bonnor W B 1957 *Mon. Not. R. Astron. Soc.* **117** 104
Dicke R H 1962 *Proc. Int. School Phys., Enrico Fermi* Course 20 (New York: Academic)
Field G and Shepley L C 1968 *Astrophys. Space Sci.* **1** 309
Hawking S W 1966 *Astrophys. J.* **145** 544
Hu B L and Regge T 1972 *Phys. Rev. Lett.* **29** 1616
Irvine W 1965 *Ann. Phys., NY* **32** 322
Johri V B 1972 *Tensor, New Ser.* **25** 241
Liang E P 1976 *Astrophys. J.* **204** 235
Perko T E, Matzner R A and Shepley L C 1972 *Phys. Rev.* D **6** 969
Silk J and Brecher K 1969 *Astrophys. J.* **158** 91